The Effects of Locality on Neural Firing Dynamics

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Introduction

The k-cap (or k-winners-take-all) process is a simple model of firing activity and inhibition in the brain. It models the formation of neural assemblies, the representation of concepts in the brain. Here we model the connectome as a directed geometric random graph in a constant dimension, which allows the synapse probability to be a function of spatial locations of the endpoints, or other neuronal features. We analyze the the dynamics of firing neurons under inhibition, as modeled by the k-cap process.

Definition of the *k*-cap Process: Given a connectome graph $G=(V,E)$ and a discrete time step t, let A_{t+1} be the set of k vertices with the **largest degree** from A_t (with ties broken randomly).
The input F_t is a function of the edges $e(y,x)$ to x:

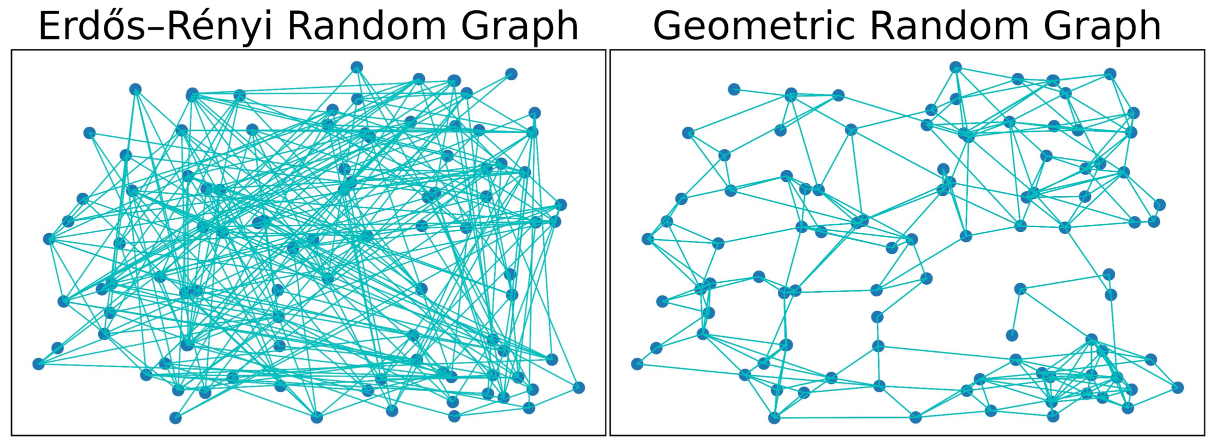
$$F_t(x) = \sum_{y \in A_t} e_{y,x}$$

And the active set is:

$$A_{t+1} = topk(\{F_t(x) : x \in V\})$$

Graph Structure:

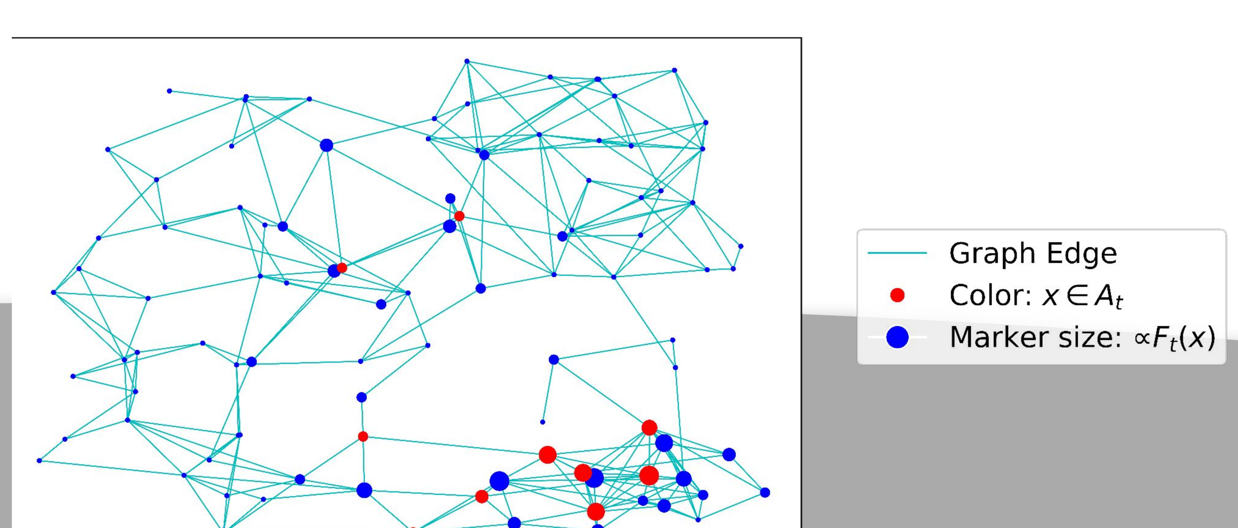
- Previous work: Erdős–Rényi random graph, $G(n,p)$-- no geometric structure, little triangle completion
- This work: Gaussian Geometric Random Graph -- accounts for locality, high triangle completion



Formally: For all $x \in V$, let $h(x) \sim U[0,1]^d$ be the *hidden variable of* x (uniform on the d-dimensional unit cube)
For all $x,y \in V$, the edge probability is:

- $G(n,p)$: $Prob((x,y) \in E) = p$, for a fixed p
- Geometric: $Prob((x,y) \in E) \propto \exp(-dist(h(x), h(y))^2/2\sigma^2)$

Example of *k*-cap on a geometric graph:



Simulations and Results

Dynamics at t=0

A_0 is chosen uniformly at random from $[0, 1]^d$.
Poisson Clumping Effect: Given uniform random points, some regions will have a higher density than average.

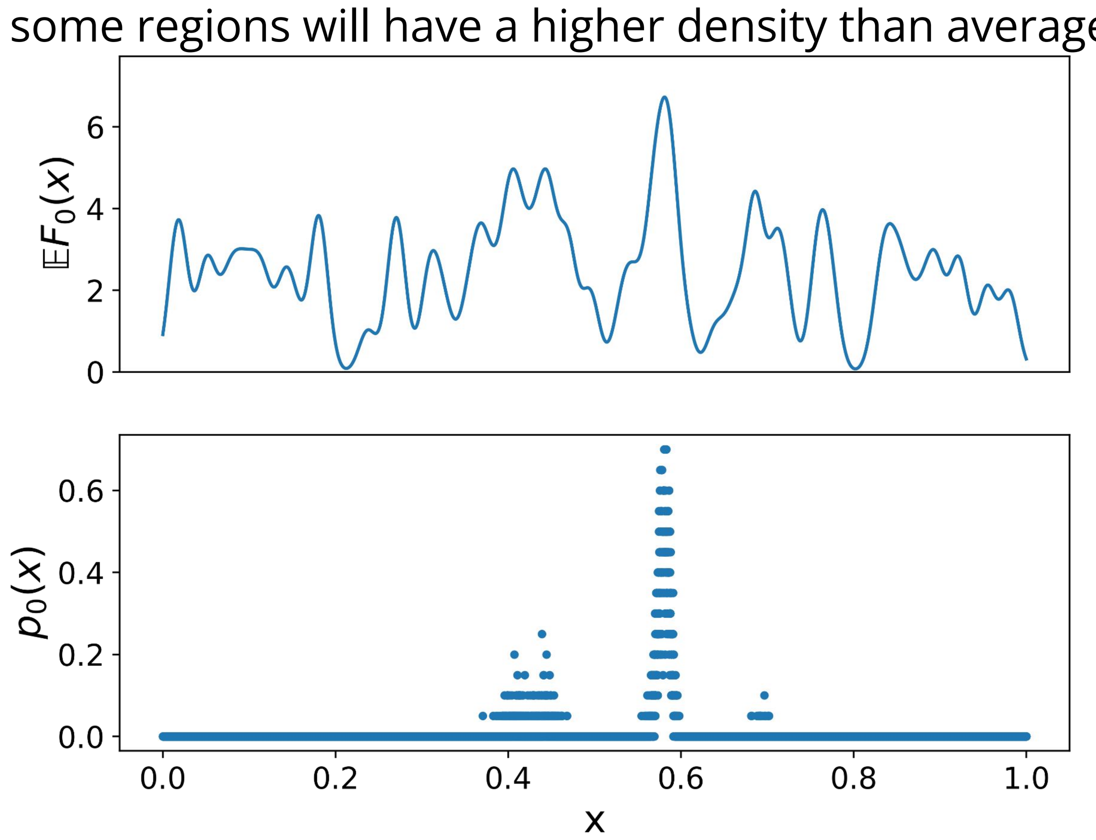


Fig: Simulation results for a 1-D graph at $t=0$. (TOP) Expected degree to x from A_0 (BOTTOM) Probability that x is in the top k. Parameters: n=10000, k=100, σ = 0.01

Parameter Range: We focus our analysis on a parameter range where Poisson Clumping emerges: Let $n=|V|$, and d be the (constant) dimension. We set: $k = n^{1/(2+d)}$, $\sigma = k^{1/d}$

Theorem: At step 1, with high probability, A_1 can be covered by $O(k^{1/4+o(1)})$ balls of size $O(\sigma \sqrt{\ln \ln k})$.

Analysis at *t>0*

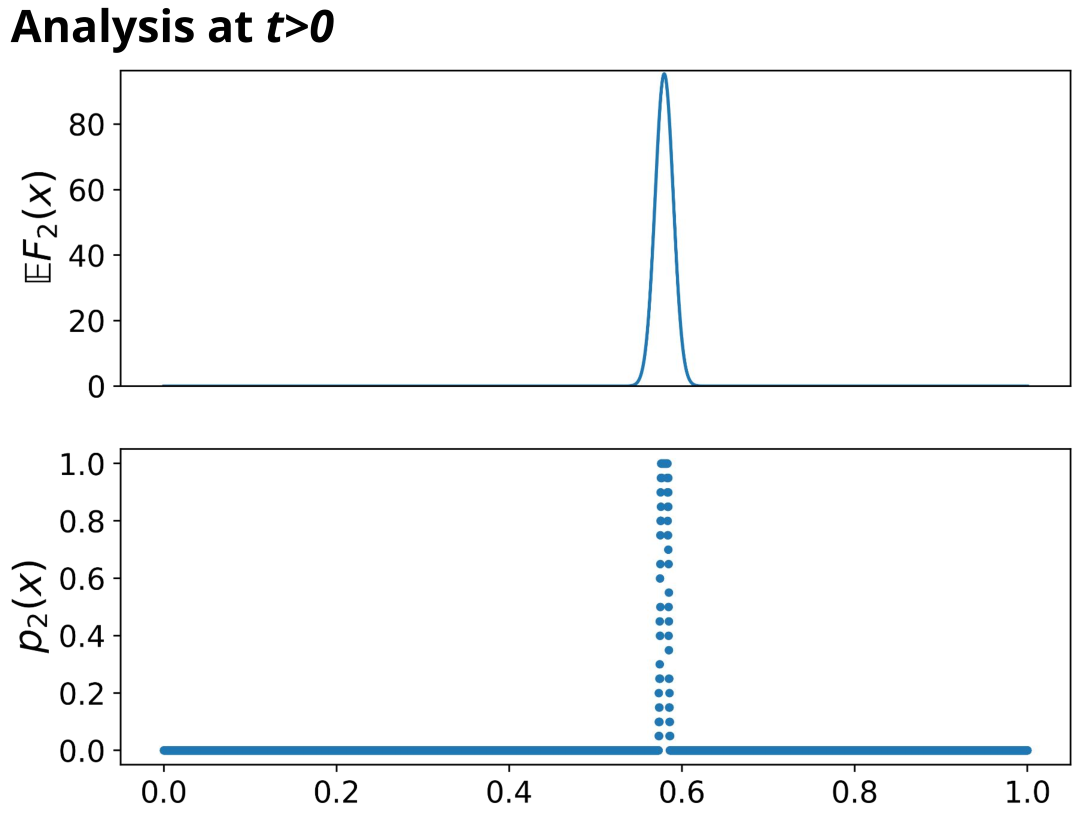
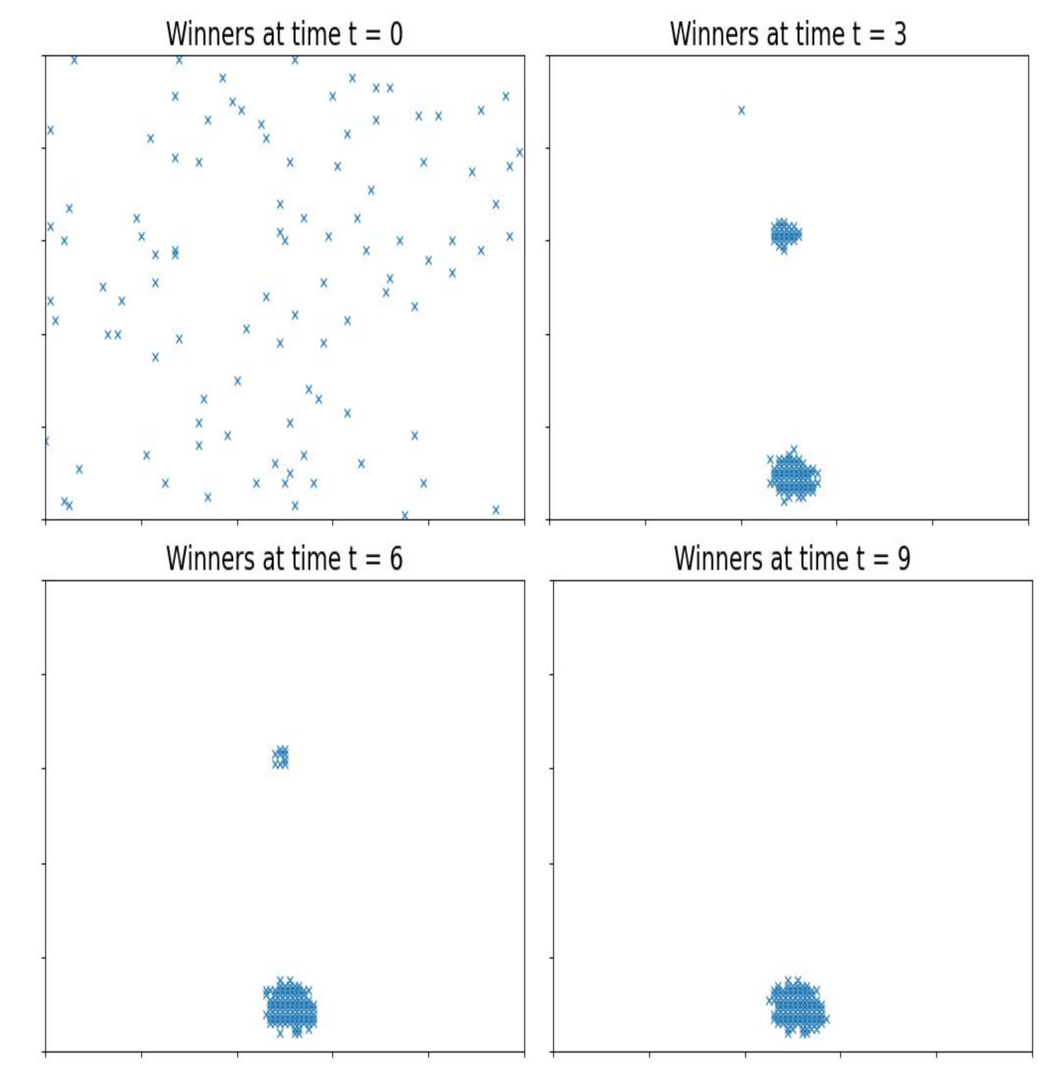


Fig: Simulation results for a 1-D graph at $t=2$. (TOP) Expected degree to x from A_2 (BOTTOM) Probability that x is in the top k. Parameters: n=10000, k=100, σ = 0.01

Simulation on a graph with hidden variables in 2D
Parameters: n=10000 vertices, k=100 cap size



Theorem: After $t^*=polylog(k)$ steps, with high probability, the winners at t^* can be covered by a single ball of size $O\left(\sigma\sqrt{\frac{\ln k}{k}}\right)$

Assembly Structure

The structure that emerges, both theoretically and in simulation, reflects properties of assemblies that have been noted experimentally but have not yet justified theoretically: our analysis reveals that within a small number of time steps ($polylog(k)$) (1) firing neurons lie in a small ball, and (2) within this ball, the subset of k firing neurons are essentially random.

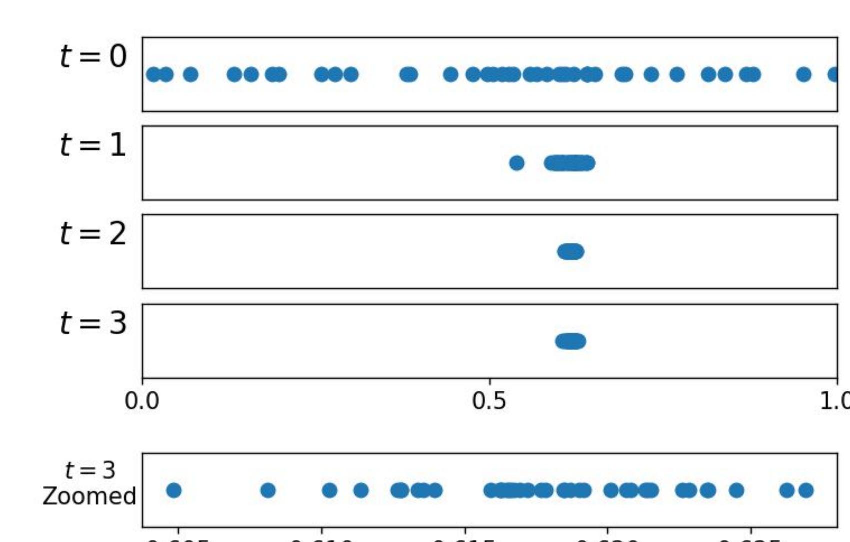


Figure: An illustration of the structure of the firing sets on a 1-D graph at different time steps. Parameters: n=90000, k=40, σ = 1/40

Results Continued

Once the k-cap process converges to a small ball, it provably remains concentrated for all t WHP

Theorem: With high probability, for all $t \geq t^*$ there exists a ball I_t with radius $r = \sigma k^{-1/3+\epsilon}$, for a constant $\epsilon>0$, such that $|A_t \cap I_t| > k - k^{2/3}$.

When high plasticity is introduced on $G(n,p)$, only $k(1+o(1))$ vertices fire over the course of the entire process. In this work, the firing set shifts randomly within a small region.

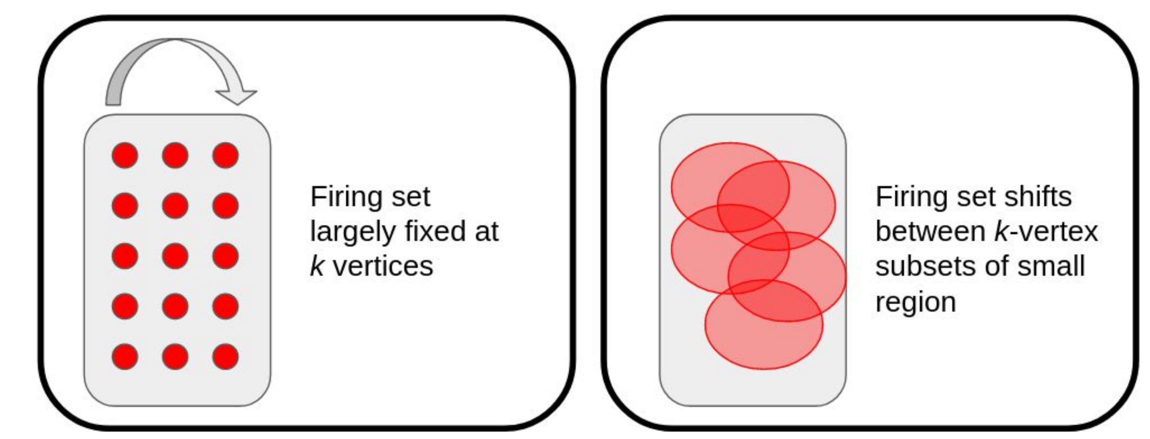


Figure: Illustration of the two notions of convergence described above: the high plasticity regime with largely fixed kcap (Left), and the low-plasticity regime where the firing set shifts randomly within a small area(right)

Conclusion

Graphs and the Brain:

- Neuronal connections are not random; probability of connection depends on distance in space.
- Geometric graphs can represent **more than just physical space**; e.g., similarity in characteristics

k-cap and the Brain:

- k-cap models an excitatory-inhibitory network with a population-wide inhibitory signal
- Adding Hebbian plasticity to the k-cap process leads to convergence on directed random graphs. This process exhibits powerful computational properties (Assembly Calculus)
- This work corresponds to a low-plasticity setting

Future Directions:

- Investigating the effect of **degree heterogeneity** on the k-cap process
  - Example: power-law graphs
- Convergence on undirected $G(n,p)$
  - Exhibits deterministic convergence to two alternating sets, but the rate is unclear.

References

Reid, Mirabel, and Santosh S. Vempala. "The $ k $-Cap Process on Geometric Random Graphs." arXiv preprint arXiv:2203.12680 (2022).
Bullmore, Ed, and Olaf Sporns. "Complex brain networks: graph theoretical analysis of structural and functional systems."
Papadimitriou, Christos H., et al. "Brain computation by assemblies of neurons."
Simulation code: https://github.com/mirabelreid/Assemblies-Simulations