The *k*-Cap Process on Geometric Random Graphs

Mirabel Reid
Georgia Tech, Dept of Computer Science
mreid48@gatech.edu

Santosh Vempala
Georgia Tech, Dept of Computer Science
vempala@cc.gatech.edu

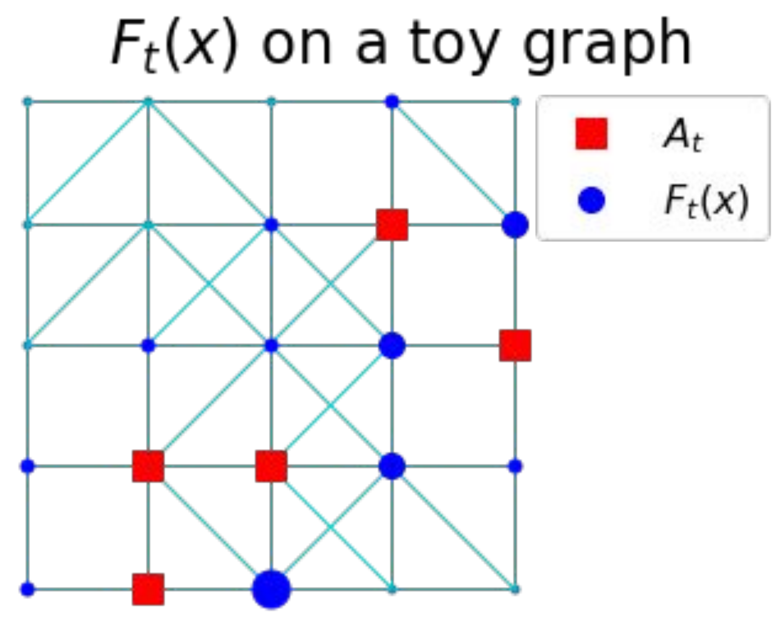
Abstract

The *k*-cap (or *k*-winners-take-all) process on a graph works as follows: in each iteration, exactly *k* vertices of the graph are in the cap (i.e., winners); the next round winners are the vertices that have the highest total degree to the current winners.
This natural process is a simple model of firing activity in the brain. We study its convergence on geometric random graphs, revealing rather surprising behavior.

Introduction

Definition of the *k*-cap Process: Given a graph *G* with *n* vertices {1, 2, ..., *n*}, at a timestep *t>0*, let A_{t+1} be the set of ***k* vertices** with the **largest degree in A_t** (with ties broken randomly). We define the input F_t as a function of the edges $e_{y,x}$ to *x*:

$$F_t(x) = \sum_{y \in A_t} e_{y,x}$$



Motivating Questions:

- *Does this process converge to a small subset of the vertices of G?*
- *If so, how quickly does it converge?*
- *How does the structure of A_t evolve as $t \to \infty$?*

Setting: Geometric Random Graphs

- Embed each vertex in a *hidden variable space*.
- If two vertices are closer in the space, they are more likely to be connected by an edge

Assumption: edge probability is proportional to 1-D *Gaussian* function with parameters $\sigma = 1/k$, $n > k^{3+\epsilon}$.

Discrete Process

A_0 chosen uniformly at random from [0, 1].
Poisson Clumping: Given uniform random points, some regions will have a higher density.

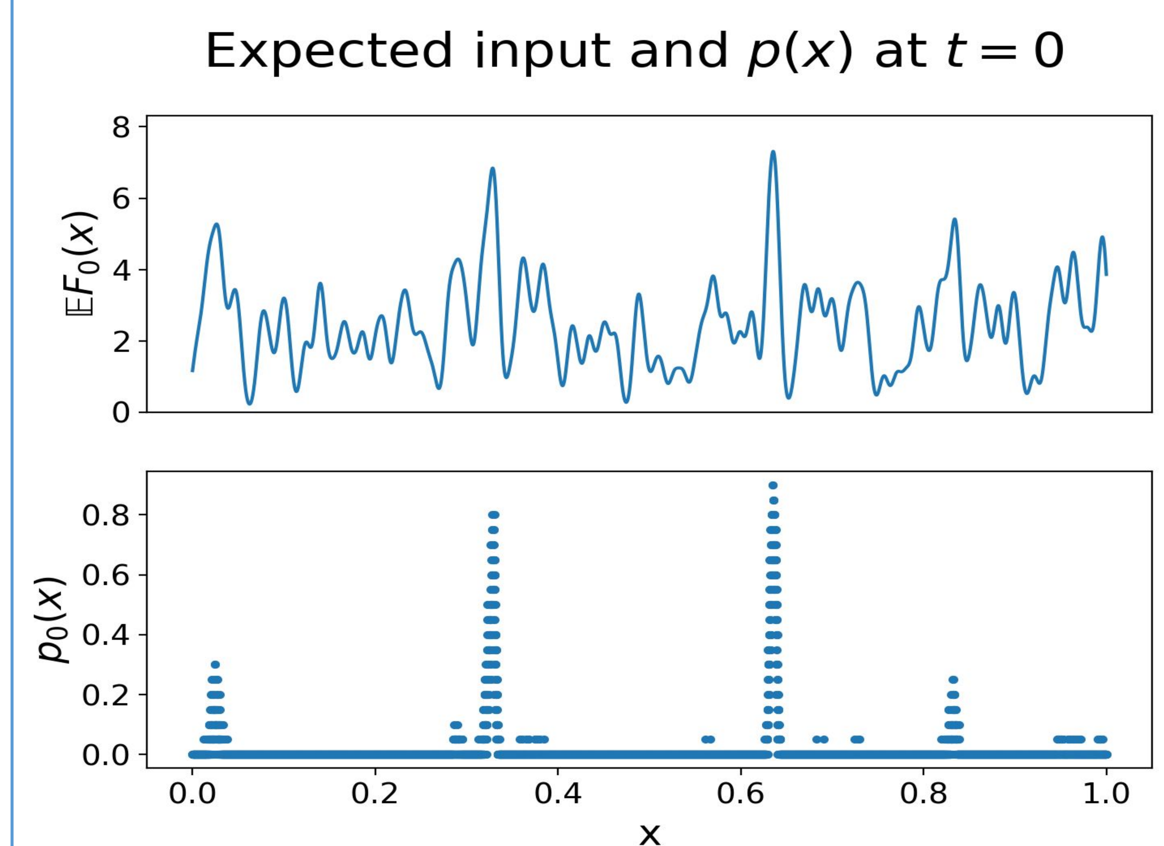


Fig:: Simulation results at *t=0*. (TOP) Expected degree to *x* from A_0 (BOTTOM) Probability that *x* is in the top *k*.

Theorem: At step 1, WHP A_1 can be covered by O(*ln k*) intervals of size O(σ *ln ln k*).

Analysis at t>1

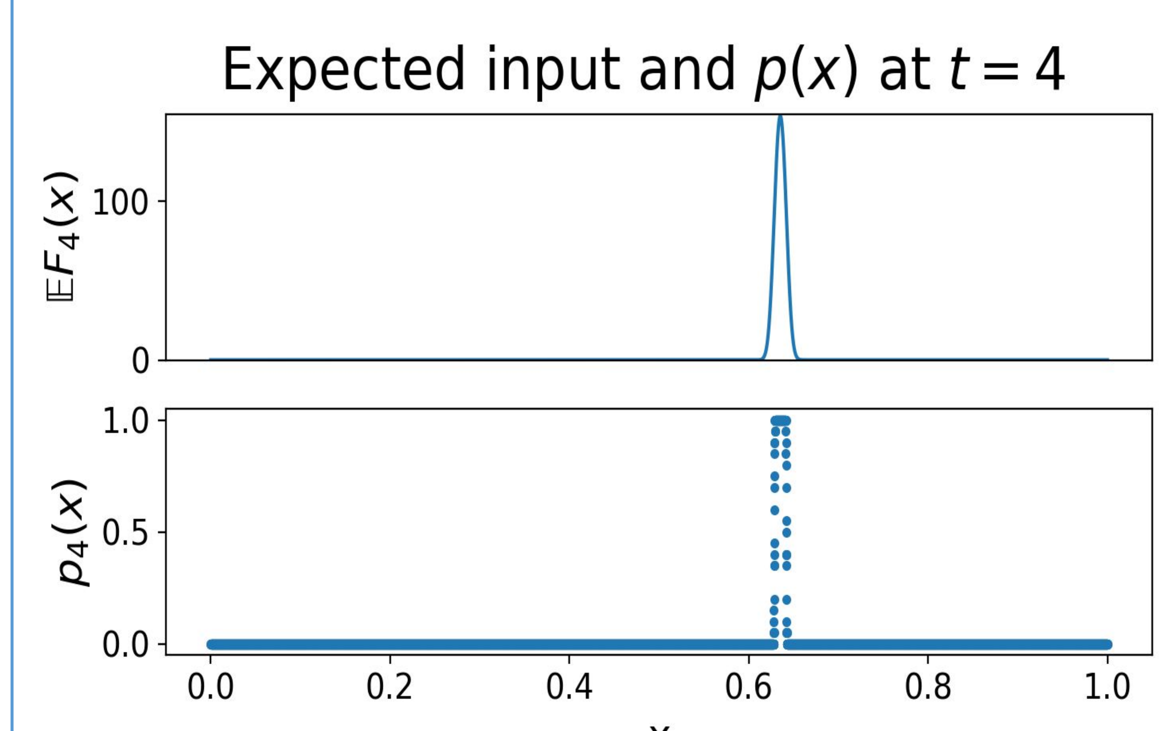
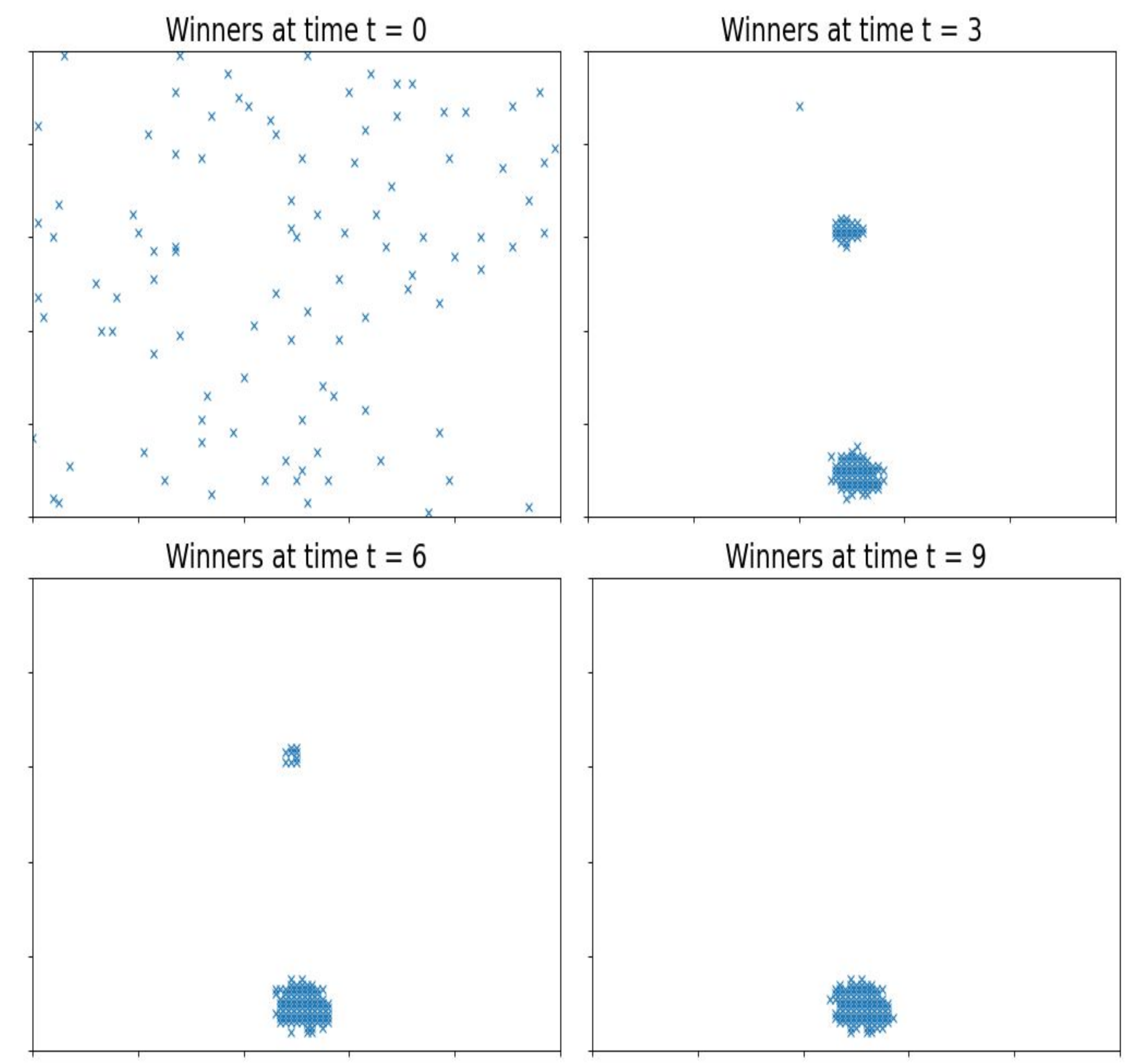


Fig: Simulation results at time t=4; showing the expected input to a vertex *x* and the probability that *x* is in the next firing set, plotted against the hidden variable.

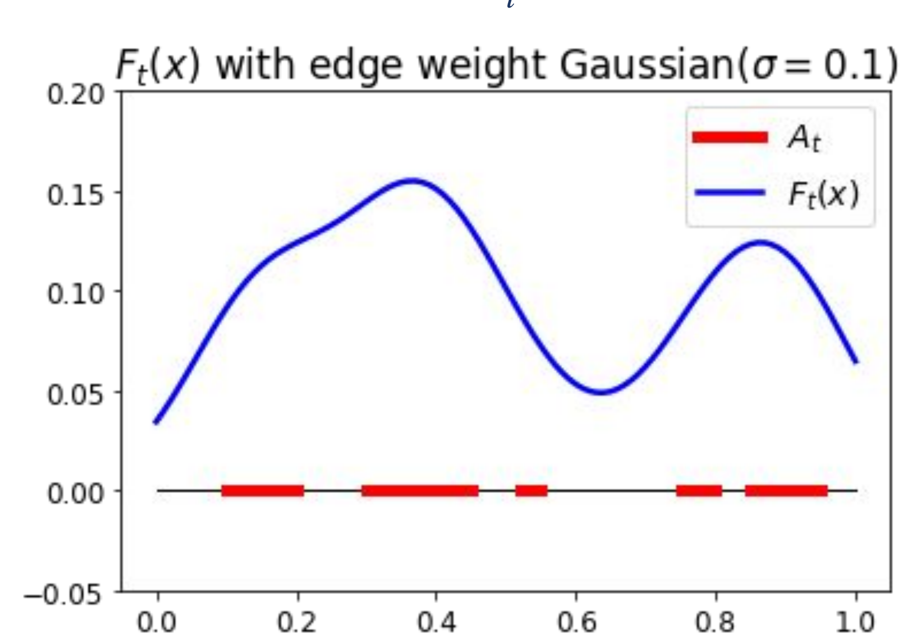
Theorem: After *polylog(k)* steps, with high probability, the winners at *t* can be covered by a **single** interval of size $O\left(\sigma\sqrt{\frac{\ln k}{k}}\right)$

Simulation on a graph with hidden variables in 2D
Parameters: *n*=10000 vertices, *k*=100 cap size



Continuous Limit: α-cap Process

Consider how the process evolves as *n* (the number of vertices in the graph) approaches ∞.
Firing set: $A_t \subset [0,1]$ is a union of intervals with measure α. An example of $F_t(x)$ is shown below:



Definition: Let $\alpha = |A_0|$ and *g* be a real integrable function. For a finite union of intervals A_t, define:

$$F_t(x) = \int_0^1 A_t(y) g(y - x)\, dy$$

$$A_{t+1}(x) = \begin{cases} 0 & F_t(x) < C_{t+1} \\ 1 & F_t(x) \geq C_{t+1} \end{cases}$$

where $C_{t+1} \in [0,1]$ *is the solution to* $\int_0^1 A_{t+1}(x) dx = \alpha$.

We showed this converges to a **single interval** of measure α.

Theorem 1. *Let A_0 be a finite set of intervals on [0,1] and g be the edge probability function. For any even, nonnegative, integrable function $g : [0,1] \to \Re_+$ with $g'(x) < 0$ for all $x > 0$, the α-cap process converges to a single interval of width α. Moreover, the number of steps to convergence is*

$$O\left(\frac{\max_{[0,1]} |g'(x)|}{\min_{[\frac{\alpha}{8},1]} |g'(x)|}\right).$$

Motivation from the Brain

Graphs and the Brain:

- We model the brain as a random graph, where neurons are vertices and synapses are edges.
- Neuronal connections are **not random**; probability of connection depends on distance in space.
- Geometric graphs have been used to model graphs of neurons;
  - Neuron position and characteristics can be represented with the hidden variable.

***k*-Cap and the Brain:**

- The **firing pattern** of neurons behaves similarly to *k*-cap;
  - **Inhibition** prevents too many neurons from firing at once.
- **Plasticity** strengthens connections between adjacent neurons.
  - Adding plasticity to the *k*-cap process leads to convergence on directed random graphs.

Future Directions:

- Extend both continuous and discrete processes to higher dimensional hidden variable spaces.
- Other graph models - e.g., convergence on *G(n,p)*

References

Reid, Mirabel, and Santosh S. Vempala. "The $ k $-Cap Process on Geometric Random Graphs." *arXiv preprint arXiv:2203.12680* (2022).
Bullmore, Ed, and Olaf Sporns. "Complex brain networks: graph theoretical analysis of structural and functional systems."
Papadimitriou, Christos H., et al. "Brain computation by assemblies of neurons."

Simulation code available at
https://github.com/mirabelreid/Assemblies-Simulations